Mathematical Writing: Proof Style Guide

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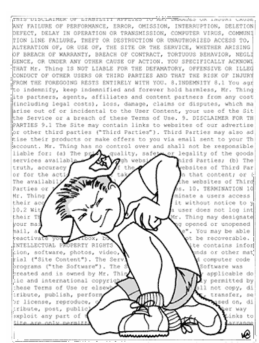
1 Introduction

In this class, you will be *writing* compositions that communicate mathematical ideas. Most of these compositions are mathematical proofs. Such compositions are so much more than just writing down a series of mathematical expressions. Indeed, a mathematical proof is intended to *convince* the reader of the correctness of a claim, and as such, includes explanations. Like any other composition, the rules of writing apply, and therefore, when writing a proof, one must abide by proper grammar, spelling, punctuation, and sentence structure.

However, it is unlikely you have had to write a composition for an English class that involved communicating mathematical ideas beyond citing statistics. Nevertheless, mathematical proofs should be written not only with mathematical correctness in mind, but also as though you were submitting one in a writing class.

2 Organize your thoughts/Use paragraphs



Short of not doing the assignment at all, the quickest way to a low score is to make your writing very difficult to read. In particular, writing a massive wall of text is the quickest way to keep someone from reading your writing. In the case of your TA and professor, writing an unorganized composition will result in a low grade. Keep in mind your TA and professors are human, and as such, fall victim to impatience.

Instead, a composition, unless very short, should be split up into multiple paragraphs. A paragraph is a **section of writing that focuses on a single idea**. When paragraphs are used properly, it is easier for a reader to find a place to take a break, take a sip of tea, come back to your composition, and easily identify where to resume reading.

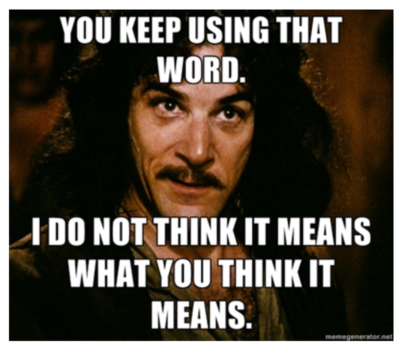
3 Know your audience

When writing a composition is knowing your audience. If you are writing a composition with a math professor as your audience, you can leave out many details such as mundane algebra and computations. However, if you were writing a composition for a peer, or even someone with slightly less mathematical background than you, you may need to include such details. You want to include just enough details so that your audience can follow along, but not so much that it bores them to tears.

However, for the purpose of this class, while you will ultimately submit your compositions to your TA or professor, your audience will be your peers. The goal is that we want you to be able to effectively communicate mathematical ideas to your peers, not just mathematicians. Incidentally, it will be highly recommended that you submit your writing to your peers for feedback, and make revisions as necessary, before submitting to your TA or professor.

4 Know what you are writing

Mathematical writing requires precision. As a general rule, you should know the precise definition of every term you use in a proof. A rather deflating experience is when a student is asked what "d divides n" means, only to receive a response of "n is divisible by d." Such a response is only a restatement of "d divides n" into passive voice, and does not actually explain what it means. Thus, a very important rule to live by when writing mathematics is that if you do not know the precise meaning of a term, **do not use it**.



Another common problem is the abuse of the word "equation" and equal sign (=). This comes in the form of students calling everything that resembles mathematical notation an equation. However, recall that an equation is a mathematical sentence in which "=" is the verb. A related abuse, of course, is using the equal sign as a generic verb.

5 Properly introduce all variables

In almost all cases, any variable that is used should be given a formal introduction. This is generally done in one of two ways:

1. When choosing an arbitrary representative of a set:

(a) Let $\varepsilon > 0$...

(b) Let n be an even integer...

(c) Let j and k be odd integers...

2. When chosen to fulfill a known quality:

   (a) Since k is odd, **there exists an integer** m **such that** $k = 2m+1$...

   (b) And so $2 = x^2$, **for some positive real number** x.

   (c) Since the function f is blurglecruncheon, **there exists a gabble-blotchit function** g **such that**...

There are notable instances when variables do not need to be explicitly introduced, but their meaning is understood from their context. Such notable instances are:

1. Variables of summation and integration:

$$\sum_{k=1}^{10} k^2, \quad \int_0^1 x^2\,dx.$$

2. References to a real vector space of arbitrary dimension: "Let $G \subseteq \mathbb{R}^n$...," in which while G is introduced explicitly as an arbitrary subset of $\mathbb{R}^n$, n is understood as being an arbitrary positive integer.

6 Sentence Structure

You may already know from English class that a sentence must contain a subject and a verb, and in some instances, an object. However, mathematics is also a language of its own, and has its parts of speech, and sentences. Consider that "$x > 2$" is in its own right a sentence, where "x" is the subject, "$>$" is the verb, and "2" is the object.

A very important part of doing mathematics is knowing when a symbolic expression represents a sentence such as "$x > 2$," "$a^n + b^n \neq c^n$," and $2 \mid 6$, and when it represents an object (a noun, if you will), such as "$x^2 + 2x + 1$."

Other rules to consider:

1. Mathematical sentences are never treated as standalone sentences in a composition, but rather as clauses, functioning as parts of sentences. For example:

   Since $x + 2 \geq 6$, it follows that $x^2 \geq 16$.

2. While mathematical sentences can be rewritten entirely in words (e.g. "x is greater than or equal to three plus y," in place of "$x \geq 3+y$"), this only makes your composition more difficult to read, and so you should use the mathematical notation to its fullest.

3. Sentences should never begin with a numeral, nor should they begin with mathematical notation. Beginning a sentence with a number is permissible if it is spelled out ("Two," "Ninety-four," etc...) and is for the purpose of counting (used in much the same way you would in any form of writing.)

4. Simple mathematical notation should be displayed inside of the paragraph as though it were just another word. Such presentation is called an in-line formula. For example:

   If $x > 3$, then $x^2 > 9$.

5. Some mathematial notation is more complex, and as such, may be difficult to display as an in-line formula. To remedy this, such notation should be displayed on its own line, in the center of the page as follows:

   If $ax^2 + bx + c = 0$, and $a \neq 0$, then:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   Notice that the quadratic formula is set in what is called a display formula. Furthermore, it is essential to remember that this formula is still considered to be part of a sentence. In this case, the quadratic formula is the end of the sentence, and so a period must be placed at the end of the formula.

6. Some formulas can also be set as a display formula if they are generally regarded as an important formula. Furthermore, it may also be convenient to number such formulas so as to be able to make references to them elsewhere in your composition. Such formula numbers can be placed in either the right or left margins, so long as you maintain a logical and easy to follow numbering system, and remain consistent.

7. An inevitable part of writing mathematical proofs is a long string of computations and algebraic manipulations. Such computations and manipulations should be set as displayed formulas, and each step is placed on its own separate line, as in:

$$\begin{aligned} a &= b \\ &= c \\ &= d. \end{aligned}$$

   Notice that all the work is done on the right-hand side, and we only right down the left-hand side on the top line. DO NOT write the left-hand side on subsequent lines; it will only serve as visual clutter. Also, notice the equal signs are lined up vertically. However, such manipulations are not limited to equalities, and in fact, other relations, such as "$\leq$" can be mixed in as follows:

$$\begin{aligned} a &= b \\ &\leq c \\ &= d \\ &< m \\ &= n. \end{aligned}$$

This, for instance, would show that $a < n$.

8. **Do not cross out** anything to signify any form of cancellation; just write the new expression on the next line.

7 Mathematical Statements

In mathematics, there are four basic classifications of statements: axioms, definitions, conjectures and theorems. Axioms are basic statements that form the basis of mathematical knowledge, and are accepted as true without proof. Definitions are statements that introduce a new term, and provide its precise meaning. Conjectures are claims for which there is no proof (and as such, are not admissible for use in a proof). Theorems are statements that are proven to be true based on axioms and previously proven theorem. It should be noted that once a proof of a conjecture is discovered, such conjecture is then considered to be a theorem.

It should also be noted that the term "theorem" itself has a couple of synonyms: "lemma" and "corollary." Use of all three are often used to distinguish importance. Mathematicians will often reserve use of the term "theorem" for only the most important and famous of results, for instance, Pythagorean Theorem, Fundamental Theorem of Calculus, and Fermat's Last Theorem. The term "lemma" is then used to denote a result that is often used as a stepping stone to a theorem. A corollary is a theorem that is very quickly proven from a previous theorem, as it is often a special case of such theorem. Choice of terms is a subjective choice, however, as indicated by the following quote from Professor Showalter:

> "One person's theorem is another person's lemma."

In mathematical writing, it may be important to include such statements. When including a mathematical statement in a composition, such statement should begin with a run-in header style label that is set in a different font style to distinguish it from the actual statement itself. Consider:

AXIOM. For all real numbers x and y, $x + y = y + x$.

DEFINITION. Let n be an integer. We say n is odd if there exists an integer k such that $n = 2k + 1$.

PROPOSITION. For every integer n, if n is odd, then n^2 is also odd.

Sometimes, it may be convenient to number mathematical statements for easy reference:

AXIOM 7.1. For all real numbers x,y and z, $x\left(y+z\right) = xy + xz$.

DEFINITION 7.2. Let n be an integer. We say n is even if there exists an integer k such that $n = 2k$.

PROPOSITION 7.3. For every integer n, if n^2 is odd, then n is also odd.

Sometimes, certain mathematical statements are well-known and important enough to earn themselves a name. Such a name can be included in the label, set in parenthesis:

THEOREM 7.4 (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for every k between $f\left(a\right)$ and $f\left(b\right)$, there exists a $c \in [a, b]$ such that $k = f\left(c\right)$.

8 Proofs

As stated before, a proof is a piece of writing that has the purpose of convincing the reader of the correctness of a theorem. When asked to prove a theorem, you will often be provided with the full statement of the theorem, and so it will be clear what the end result should be, although once in a while, you may be tasked with proposing a statement of a theroem. When assigned a problem set, a cue for when a proof is appropriate is when such a problem begins with "Show that..." or "Prove that..."

1. A proof should always be preceeded with a precise statement of the theorem that is to be proven.
2. Use run-in header style to denote the beginning of the proof, and QED or other decoration (usually a square) to denote the end of the proof.
3. Use *first person plural* pronouns, as if both the author and reader are proving the result together.
4. Unless method of proof is direct, begin the proof with brief introductory paragraph that conveys the method of proof that is used.
5. For multipart proofs, use a run-in header to clearly mark each section.

(a) For proofs of biconditional statements ($P \iff Q$), one part is headed with "$\implies$" for the proof of $P \implies Q$, and another part is headed with "$\impliedby$" for the reverse implication $Q \implies P$. For example:

Proposition. Let S be a finite set. A function $f : S \to S$ is one to one if and only if it is onto.

Proof. ($\implies$.) Suppose f is one-to-one. We will show f is onto.
...

($\impliedby$.) Suppose f is onto. We will show f is one-to-one.
... ❑

(b) Induction proofs have a "Basis" part and "Induction" part.

Proposition. All cars are of the same color.

Proof. We will prove this result using induction on the number of cars in a set.

(*Basis.*) In a set of **one** car, surely all cars are of the same color.

(*Induction.*) Now let $n \in \mathbb{N}$ and suppose all cars in a set of n are of the same color. We will now show that all cars in a set of $n + 1$ are of the same color.
... ❑

(c) Proofs by cases have parts labeled "Case 1," "Case 2," etc..

(d) Each part in a multipart proof should generally begin with an introduction clearly stating what is being assumed and, with exception to proof by cases, what is to be proven.

Stuck on how to begin? Consider the following templates for how to begin the following types of proofs.

8.1 Direct proof

A direct proof is a proof of a conditional statement:

$$\forall x \in U : \underbrace{P(x)}_{\text{Hypothesis}} \implies \underbrace{Q(x)}_{\text{Conclusion}} .$$

In such a proof, we first choose an arbitrary representative of the domain ("Let $x \in U$..."), and then we assume the hypothesis. From there, we prove the conclusion. We may begin such a proof utilizing the following template:

Proof. Let $x \in U$ and assume $P(x)$. We will now show $Q(x)$.

... ❑

Keep in mind this is a *template*, and as such, we **do not just write** "$P(x)$" and "$Q(x)$," but rather write out the propositional function, as in the following example:

Proposition. For each function $f : \mathbb{R} \to \mathbb{R}$, if f is globberfluxible, then f is odd.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and assume f is globberfluxible. We will now show f is odd.

... ❑

8.2 Proof by contrapositive

Once in a while, you may encounter the task of having to prove a conditional statement:

$$\forall x \in U : P(x) \implies Q(x),$$

but find that direct proof does not work. In particular, there may not be an easy logical argument that deduces $Q(x)$ when one assumes $P(x)$.

The good news, however, is that the above conditional statement is logically equivalent to its contrapositive:

$$\forall x \in U : \neg Q(x) \implies \neg P(x),$$

and applying the method of direct proof to the contrapositive will work. This gives the following template:

Proof. We will prove the contrapositive. Let $x \in U$ and assume $\neg Q(x)$. We will now show $\neg P(x)$.

... ❑

Keep in mind this is a *template*, and as such, we do not just write "$P(x)$" and "$Q(x)$," but rather write out the actual clauses, as in the following example:

Proposition. For each $n \in \mathbb{Z}$, if n is globberfluxible, then n is odd.

Proof. We will prove the contrapositive. Let $n \in \mathbb{Z}$, and assume n is even. We will now show n is not globberfluxible.

... ❑

Notice in this instance that it will be preferred to say "even" instead of "not odd." Had there been a term that is precisely equivalent to "not globberfluxible," it would be preferred that we use that term instead.

8.3 Proof by contradiction

Some statements can only be proven by contradiction, in which it is assumed the statement is false, with the goal of deducing an absurdity, either in the form of showing another statement to be both true *and* false simultaneously, or proving a result that goes against an axiom or previously-established theorem. Typically, when one writes a contradiction proof, one uses language that emphasises the fact that one is speaking hypothetically ("If $\sqrt{2}$ were rational, then there would exist..." instead of "Since $\sqrt{2}$ is rational, there exist...").

If we were to prove a statement P, a proof by contradiction may begin as follows:

Proof. Suppose otherwise, that $\neg P$. We will deduce a contradiction.

... ❑

A famous example of a proof by contradiction is the proof that there are infinitely many prime numbers.

Proof. Suppose otherwise, that there are only finitely many primes, say p_1, p_2, ..., p_n. We will deduce a contradiction.

... ❑

For those interested, the rest of the proof involves multiplying all these prime numbers together (something you cannot do to an infinite number of primes), and go on to eventually show that 1 would have a prime divisor, something that we know to be false.

It should be noted that applying proof by contradiction to a conditional statement if often a proof by contrapositive in disguise.

8.4 Induction

Proof by inducton is perhaps the most difficult proof method to master, and the difficulty is generally seen in the opening paragraph of the Induction part. Recall that the axiom of induction can be written down in the following symbolic form:

$$\left(\underbrace{P\left(n_0\right)}_{\text{Basis}} \wedge \underbrace{\forall k \geq n_0 : P\left(k\right) \implies P\left(k+1\right)}_{\text{Induction}}\right) \implies \forall n \geq n_0 : P\left(n\right).$$

Note that the Induction part is itself proof of the conditional statement:

$$\forall k \geq n_0 : P\left(k\right) \implies P\left(k+1\right),$$

and as such, the template for direct proofs applies as follows:

(*Induction.*) Let $k \geq n_0$ and assume $P(k)$. We will now show $P(k+1)$.

What **not to do** (common mistakes encountered when grading):

(*Induction.*) Assume $P(k)$ for all $k \geq n_0$. We will now show $P(k+1)$.

Remember to follow the basic rule that k is introduced *first* as an arbitrary representative of the domain of discourse, *then* we state the assumption. Assuming "$P(k)$ for all $k \geq n_0$" is not the same as assuming $P(k)$ at an arbitrary choice of k. In fact, "$P(k)$ for all $k \geq n_0$" precisely the statement we are attempting to prove, so we would be begging the question.

Again, as before, we *write out* what $P(k)$ and $P(k+1)$ are.

(*Induction.*) Let $k = k+1$. We will now show $P(k+1)$.

Writing "$k = k+1$" is itself an absurdity. There is *no real value* of k whatsoever that makes this happen.

9 Epilogue

Writing mathematical proofs is not something that is quickly mastered, but it can be a rewarding experience. Remember that the goal of a proof is to convince the reader of the correctness of a statement. Furthermore, mathematics is meaningless without a way to communicate its ideas, and so it is prudent to learn how to properly integrate (pun not intended) mathematics with writing. Thus, it is a good idea to do what is generally encouraged with any piece of writing, including peer review. After all, you want to make sure your peers completely understand your writing, as well as your logical argument. Furthermore, it is always a good idea to read back (aloud) your own writing to yourself.

Finally, one last piece of advice: Do your computations on scratch paper first, especially if you are attempting a computation, and are not sure if it will lead you to your desired result. Once you arrive at a valid computation that contributes to your proof, then you can clean up the computation if necessary, and include it in your final submission.