2.2 Distributions

In this section, we shall take $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, both open.

The theory of distributions is due to Laurent Schwartz[2]. The presentation we give here follows that of Lars Hörmander[1]. The idea of distributions is to generalize the notion of functions to permit analysis that is not possible in the classical setting. For instance, differential operators are not defined on $\mathcal{L}^p(X)$, yet $\mathcal{C}^\infty(X)$, the space of functions for which differential operators may be applied arbitrarily many times, is too small of a class of functions. Even smaller is $\mathcal{C}_0^\infty(X)$, the space of functions in $\mathcal{C}^\infty(X)$ with compact support, and yet, $\mathcal{C}_0^\infty(X)$ offers enough flexibility to be used as a space of test functions in that given any $f \in \mathcal{L}^1_{\text{loc}}(X)$, then $\mathcal{T}_f$, defined by

$$\mathcal{T}_f(\phi) = \int_X f(x)\,\phi(x)\;dx, \quad \phi \in \mathcal{C}_0^\infty(X), \tag{2.2.1}$$

defines a linear form on $\mathcal{C}_0^\infty(X)$. Furthermore, the mapping $f \mapsto \mathcal{T}_f$ is one-to-one, modulo functions that differ on a set of measure zero. Therefore, the algebraic dual $\mathcal{C}_0^\infty(X)^\star$ contains an embedding of $\mathcal{L}^1_{\text{loc}}(X)$, and therefore also contains an embedding of $\mathcal{L}^p(X)$, for $1 \le p \le \infty$. Furthermore, if f is itself differentiable in at least the weak sense, then integration by parts tells us that

$$\mathcal{T}_{\mathcal{D}_{\vec{\mathbf{v}}} f}(\phi) = -\mathcal{T}_f(\mathcal{D}_{\vec{\mathbf{v}}}\phi),$$

which inspires defining $\mathcal{D}_{\vec{\mathbf{v}}}$ as an operator on $\mathcal{C}_0^\infty(X)^\star$ by

$$\mathcal{D}_{\vec{\mathbf{v}}} u(\phi) = -u(\mathcal{D}_{\vec{\mathbf{v}}}\phi), \quad u \in \mathcal{C}_0^\infty(X)^\star, \phi \in \mathcal{C}_0^\infty(X). \tag{2.2.2}$$

Unfortunately, $\mathcal{C}_0^\infty(X)^\star$ offers no topology, and in fact, is too large for analysis. We will want to restrict to a subspace of $\mathcal{C}_0^\infty(X)^\star$ that has a topology in which $\mathcal{C}_0^\infty(X)$ is embedded, and is dense, yet is not too small. Such a topology would not come from a norm. Hörmander proposes the following:

Definition 2.2.0.1 (Hörmander 2.1.1). *A distribution u in X is a linear form on $\mathcal{C}_0^\infty(X)$ such that for every compact set $K \subseteq X$, there exist constants C and k such that*

$$|u(\phi)| \le C \sum_{|\alpha| \le k} \sup_{x \in K} |\partial^\alpha \phi(x)|, \quad \phi \in \mathcal{C}_0^\infty(K). \tag{2.2.3}$$

The set of all distributions in X is denoted by $\mathcal{D}'(X)$. If the same integer k can be used in (2.2.3) for every K, we say that u is of order $\leq k$, and denote the set of such distributions by $\mathcal{D}'^k(X)$. Their union $\mathcal{D}'_F(X) = \bigcup_{k=0}^{\infty} \mathcal{D}'^k(X)$ is the space of distributions of finite order.

For $1 \leq p \leq \infty$ and $k \geq 0$, we define the Sobolev space $\mathcal{W}^{k,p}(X)$ to be the space of functions $f \in \mathcal{L}^p(X)$ possessing all k-th order weak derivatives in $\mathcal{L}^p$. In particular, for each mutliindex α with $|\alpha| \leq k$, there is a function $\partial^\alpha f \in \mathcal{L}^p(X)$ for which

$$\int_X f(x)\, \partial^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_X \partial^\alpha f(x)\, \phi(x)\, dx, \quad \phi \in \mathcal{C}_0^\infty(X).$$

One choice of norm on $\mathcal{W}^{k,p}(X)$ is then given by

$$\|f\|_{\mathcal{W}^{k,p}} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{\mathcal{L}^p},$$

and with this norm, $\mathcal{W}^{k,p}(X)$ is a Banach space that is continuously embedded in $\mathcal{L}^p(X)$.

In particular, $u \in \mathcal{C}_0^\infty(X)^\star$ is a distribution if for every compact subset K of X, $u|_{\mathcal{C}_0^\infty(K)}$ is a continuous linear form on $\mathcal{C}_0^\infty(K)$ under some $\mathcal{W}^{k,\infty}$ norm.

We use the weak-* topology on $\mathcal{D}'(X)$. That is, we say $u_k \to u$ in $\mathcal{D}'(X)$ if $u_k(\phi) \to u(\phi)$ for all $\phi \in \mathcal{C}_0^\infty(X)$.

If $u \in \mathcal{D}'(\mathbb{R}^n)$ additionally satisfies the estimate

$$|u(\phi)| \leq C_\beta \sum_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} \left|x^\beta \partial^\alpha \phi(x)\right|, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n), \tag{2.2.4}$$

for all multi-indices β, then u extends continuously to $\mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions, defined as

$$\mathcal{S}(\mathbb{R}^n) = \left\{ \phi \in \mathcal{C}^\infty(\mathbb{R}^n) \,\middle|\, \sup_{x\in\mathbb{R}^n} \left|x^\beta \partial^\alpha \phi(x)\right| < \infty, \forall \alpha, \beta \right\}.$$

Continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ are called **tempered distributions**, and we denote the space of such linear functionals as $\mathcal{S}'(\mathbb{R}^n)$.

The utility of estimate (2.2.3) becomes clear in the proof of the following theorem from Hörmander:

Theorem 2.2.0.1 (Hörmander 2.1.3)**.** *If $\phi \in \mathcal{C}^{\infty}(X \times Y)$, where Y is an open set in $\mathbb{R}^m$, and if there is a compact set $K \subseteq X$ such that $\phi(x, y) = 0$ when $x \notin K$, then*

$$y \mapsto u(\phi(x, y))$$

is a $\mathcal{C}^{\infty}$ function of y if $u \in \mathcal{D}'(X)$ and

$$\partial_y^{\alpha} u(\phi(x, y)) = u\left(\partial_y^{\alpha} \phi(x, y)\right).$$

The proof of this result can be seen by treating ϕ as smooth function of y, taking values in $\mathcal{C}_0^{\infty}(X)$. More specifically, if we let k be the order of u on K, we can identify ϕ with an infinitely differentiable map, in the Fréchet sense, from Y into $\mathcal{W}^{k,\infty}(X)$, an argument which requires verifying that for each $y_0 \in Y$, $\partial^{\alpha}\phi(x, y) \to \partial^{\alpha}\phi(x, y_0)$ *uniformly* in x as $y \to y_0$ for each multiindex α. The result then follows via chain rule, using the continuity requirement of u.

The notation used in the above theorem is due to Hörmander, however, we will hereafter use the notation $u_x(\phi(x, y))$ in place of $y \mapsto u(\phi(x, y))$ to denote that u is acting on ϕ as a function of x only, holding y as a parameter. Do notice that can slightly weaken the support requirement on ϕ specified in Theorem 2.2.0.1 by only requiring that each $y_0 \in Y$ has an open neighborhood to which the restriction of ϕ still satisfies the above support requirement.

The usefulness of this result then becomes clear when we see that if ϕ is itself in $\mathcal{C}_0^{\infty}(X \times Y)$, then $u_x(\phi(x, y)) \in \mathcal{C}_0^{\infty}(Y)$. This gives meaning to the formula $w_y(u_x(x, y))$ when $w \in \mathcal{D}'(Y)$. In fact, in view of Theorem 5.1.1 of Hörmander, this very formula defines a distribution on $X \times Y$, which we call the **tensor product** of u and w, denoted by $u \otimes w$; it is furthermore equal to $u_x(w_y(\phi(x, y)))$. The equality of these two formulas is easily seen as an application of Fubini-Tonelli when both u and w are identified with $\mathcal{L}^1_{\text{loc}}$ functions.

As it turns out, beyond the linear structure, we can carry over some ideas from classical analysis over to the theory of distributions.

2.2.1 Support of a distribution

Motivated by the fact that $\int_X f(x)\,\phi(x)\,dx = 0$ for every $\phi \in \mathcal{C}_0^{\infty}(X \backslash \text{supp}(f))$, and $\text{supp}(f)$ is the smallest closed set satisfying such property, we can easily extend the definition of support to distributions. Hörmander first defines the restriction of a distribution u to an open set $Y \subseteq X$ as being the restriction (in the usual sense) to $\mathcal{C}_0^\infty(Y)$, before the definition of support as follows:

Definition 2.2.1.1 (Hörmander 2.2.2)**.** *If $u \in \mathcal{D}'(X)$, then the support of u, denoted* $\operatorname{supp}(u)$*, is the set of points in X having no open neighborhood to which the restriction of u is 0.*

In other words, the complement of $\operatorname{supp}(u)$ is the largest open subset $U \subseteq X$ for which $u(\phi) = 0$ for all $\phi \in \mathcal{C}_0^\infty(U)$. We define $\mathcal{E}'(X)$ as being the set of compactly-supported distributions on X. This notation is due to Schwartz defining $\mathcal{E}(X)$ as the space of $\mathcal{C}^\infty$ functions with the topology defined by the family of seminorms

$$\phi \mapsto \sum_{|\alpha| \le k} \sup_{x \in K} |\partial^\alpha \phi(x)|, \quad k \ge 0, K \subset\subset X.$$

Theorem 2.3.1 from Hörmander then states that the space of compactly-supported distributions is identical to the dual space of $\mathcal{E}(X)$. While in general, $\mathcal{D}'(X)$ cannot be embedded in $\mathcal{D}'(\mathbb{R}^n)$, we can embed $\mathcal{E}'(X)$ inside $\mathcal{D}'(\mathbb{R}^n)$ in a natural way, and as such, any operators on $\mathcal{D}'(\mathbb{R}^n)$ can act via this embedding on $\mathcal{E}'(X)$.

The singular support of u, denoted as $\operatorname{sing\,supp}(u)$, is defined as the smallest closed set D for which u is given by a $\mathcal{C}^\infty$ function on $X \backslash D$. That is, there exists $f \in \mathcal{C}^\infty(X \backslash D)$ such that

$$u(\phi) = \int_{X \backslash D} f(x)\, \phi(x)\; dx, \quad \phi \in \mathcal{C}_0^\infty(X \backslash D).$$

While distributions are defined as linear forms on $\mathcal{C}_0^\infty$, distributions with support restrictions offer flexibility to be defined as a linear form on a larger space than $\mathcal{C}_0^\infty$. For instance, in the case that a distribution u is compactly-supported, it is nearly trivial to show that $u(\phi)$ can be defined for *any* $\phi \in \mathcal{C}^\infty(X)$, by taking $u(\phi) = u(\psi\phi)$ for some $\psi \in \mathcal{C}_0^\infty(X)$ that is equal to 1 on an open neighborhood of $\operatorname{supp}(u)$. Such an extension will be unique, as an alternate choice for ψ can be shown to yield the same extension.

Even if u is not compactly supported, it will be similarly possible to define $u(\phi)$ from only the condition that $\operatorname{supp}(\phi) \cap \operatorname{supp}(u)$ is compact by defining

$u(\phi)$ as above, but instead taking ψ equal to 1 on an open neighborhood of $\operatorname{supp}(\phi) \cap \operatorname{supp}(u)$. Consequently, this allows us to give meaning to $w_y(u_x(\phi(x,y)))$ for $\phi \in \mathcal{C}^\infty(X \times Y)$, when $\operatorname{supp}_y(u_x(\phi(x,y))) \cap \operatorname{supp}(w)$ is compact, as is $\operatorname{supp}_x(\phi(x,y)) \cap \operatorname{supp}(u)$ for all $y \in Y$.

2.2.2 Multiplication

From formula (3.1.2) of Hörmander, one can multiply a distribution $u \in \mathcal{D}'(X)$ by a smooth function $\psi \in \mathcal{C}^\infty(X)$ by defining $(\psi u)(\phi) = u(\psi\phi)$. It is easy to extend to the case that ψ is only $\mathcal{C}^\infty$ on an open neighborhood of $\operatorname{supp} u$ by replacing ψ with a function in $\mathcal{C}^\infty(X)$ that is equal to ψ on $\operatorname{supp}(u)$, which then allows us to define the quotient $\frac{u}{\psi}$ when $\psi \in \mathcal{C}^\infty(X)$ is nonzero on $\operatorname{supp}(u)$.

In general, multiplication of two distributions is not defined. However, two distrubtions may have a well-defined product under some conditions. For instance, if two distributions have disjoint singular supports, then their product can be defined. Later, wavefront set analysis will help us establish a weaker condition when the product of two distributions can indeed be defined, thereby extending multiplication of distributions past the case of disjoint singular supports.

2.2.3 Composition with smooth functions

If Φ is a diffeomorphism from X to Y, then for $f \in \mathcal{L}^1_{\mathrm{loc}}(Y)$, one has

$$\int_X f(\Phi(x))\,\phi(x)\,dx = \int_Y f(y)\,\phi\left(\Phi^{-1}(y)\right)\left|\det \mathcal{D}\Phi^{-1}(y)\right|\,dy, \quad \phi \in \mathcal{C}_0^\infty(X),$$

and so for $u \in \mathcal{D}'(Y)$, we define $\Phi^\star u \in \mathcal{D}'(X)$ by

$$\Phi^\star u(\phi) = u\left(\left|\det \mathcal{D}\Phi^{-1}\right| \cdot \Phi^{-\star}\phi\right), \quad \phi \in \mathcal{C}_0^\infty(X).$$

If we only require Φ be a local diffeomorphism, then we can similarly define $\Phi^\star u$ with localization methods. Additionally, if Φ is a surjection onto Y, then $\Phi^\star$ is a one-to-one map from $\mathcal{D}'(Y)$ into $\mathcal{D}'(X)$. Thus, even if Φ is not globally invertible, we may define $\Phi^{-\star}w$ for $w \in \mathcal{D}'(Y)$ as being the distribution $u \in \mathcal{D}'(X)$ satisfying $w = \Phi^\star u$, if such a distribution exists. If a distribution w is in the range of $\Phi^\star$, then one can verify that for any pair of open sets U and V of X for which the restrictions $\Phi|_U$ and $\Phi|_V$ are injective, and $\Phi(U) = \Phi(V)$, we have

$$w(\phi) = w\left(\left|\det \mathcal{D}\Psi^{-1}\right| \cdot \Psi^{-\star}\phi\right), \quad \phi \in \mathcal{C}_0^\infty(U),$$

where $\Psi = \Phi|_V^{-1} \circ \Phi|_U : U \to V$ is the induced transition map. As it turns out, this condition is also sufficient for w to be in the range of $\Phi^\star$, and $\Phi^{-\star}w$ can also be constructed using localization methods.

The usefulness of reversing a pullback comes when working in an alternate coordinate system. For example, if a distribution $w \in \mathcal{D}'(X)$, where

$$X = \{(r,\theta) \,|\, r > 0, \theta \in \mathbb{R}\},$$

is 2π periodic in θ – that is, $w_{(r,\theta)}(\psi(r, \theta + 2\pi)) = w(\psi)$ for $\psi \in \mathcal{C}_0^\infty(X)$, then w can be seen as a distribution in polar coordinates, and be pushed back to a distribution $u = \Phi^{-\star}w \in \mathcal{D}'\left(\mathbb{R}^2 \backslash 0\right)$ in Cartesian coordinates, where $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$.

Hörmander takes composition further with Theorem 6.1.2 by defining $\Phi^\star u$ when $\mathcal{D}\Phi(x)$ is merely surjective for every $x \in X$. This, of course, requires $n \geq m$. Such a construction requires we find a smooth function $\Psi : X \to \mathbb{R}^{n-m}$ so that $\Phi \oplus \Psi$ is a local diffeomorphism, and then we define

$$\Phi^\star u = (\Phi \otimes \Psi)^\star (u \otimes 1),$$

where 1 is identified as the constant map defined on $\mathbb{R}^{n-m}$.

In the case that $m > n$, $\Phi^\star u$ may be defined in some cases, but requires analysis of wavefront sets.

2.2.4 Convolution

When $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, the convolution $f \star \phi$ is defined by

$$f \star \phi(x) = \int_{\mathbb{R}^n} f(y)\,\phi(x-y)\,dy,$$

and so in keeping with this formula, Hörmander defines the convolution $u \star \phi$ between $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ in section 4.1 with

$$u \star \phi(x) = u_y(\phi(x-y)).$$

Theorem 4.1.1 from Hörmander establishes that this function is $\mathcal{C}^\infty(\mathbb{R}^n)$, with $\mathcal{D}_{\vec{\mathbf{v}}}(u \star \phi) = \mathcal{D}_{\vec{\mathbf{v}}} u \star \phi = u \star \mathcal{D}_{\vec{\mathbf{v}}} \phi$, which follows immediately from Theorem 2.2.0.1. Furthermore, we also have $\text{supp}(u \star \phi) \subseteq \text{supp}(u) + \text{supp}(\phi)$. Theorem 4.1.2 also establishes that convolution with a smooth function also satisfies an associative property, in the sense that if we also have $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then $(u \star \phi) \star \psi = u \star (\phi \star \psi)$.

In Theorem 4.2.1, Hörmander then establishes that for $v, w \in \mathcal{E}'(\mathbb{R}^n)$, if either v or w has compact support, a unique distribution u exists satisfying

$$u \star \phi = v \star (w \star \phi), \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n), \tag{2.2.5}$$

which is then defined as the convolution of v and w, denoted by $u = v \star w$. In particular, by resolving $u \star \{\phi(-x)\}$, we recover an explicit formula for the convolution of two distributions:

$$u(\phi) = v_x(w_y(\phi(x+y))). \tag{2.2.6}$$

This formula is found in Chapter 5 of Schwartz. Theorem 4.1.5 of Hörmander then establishes by construction through convolutions that $\mathcal{C}_0^\infty(X)$ is dense in $\mathcal{D}'(X)$ with the weak-* topology.

The Dirac distribution, δ, is defined by $\delta(\phi) = \phi(0)$. It is a quick computation to verify that $\delta \star w = w \star \delta = w$ for all $w \in \mathcal{D}'(\mathbb{R}^n)$.

While Hörmander requires either v or w to be compactly-supported to define $v \star w$, Schwartz weakens the support restriction in Chapter 6, Section 5, to only require that $(K - \text{supp}(w)) \cap \text{supp}(v)$ be compact for every compact set K. This weaker support condition ensures that for any $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\text{supp}_x(w_y(\phi(x+y))) \subseteq \text{supp}(\phi) - \text{supp}(w),$$

has a compact intersection with $\text{supp}(v)$.

It should be observed that it is *not enough* for $(\{x_0\} - \text{supp}(v)) \cap \text{supp}(w)$ to be compact for every $x_0 \in \mathbb{R}^n$, as the following example illustrates:

Example 2.2.4.1. *Consider the distribution* w *on* $\mathbb{R}^2$*, defined by*

$$w(\phi) = \int_{\mathbb{R}} \phi(x, e^x)\, dx, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

If we were to convolve w *with* $J_{\vec{\mathbf{e}}_1}$*, where* $\vec{\mathbf{e}}_1 = (1,0)$ *in the* xy *plane, we would consider that the support of* $(\{x_0\} - \operatorname{supp}(w)) \cap \operatorname{supp}(J_{\vec{\mathbf{e}}_1})$ *is at most a single point, but* $(K - \operatorname{supp}(w)) \cap \operatorname{supp}(J_{\vec{\mathbf{e}}_1})$ *fails to be compact if* K *contains some open set intersecting the* x*-axis. The convolution, according to* (2.2.6)*, would then be*

$$\begin{aligned} w \star J_{\vec{\mathbf{e}}_1}(\phi) &= w_{(x,y)}\left(\int_0^\infty \phi(x+t, y)\, dt\right) \\ &= \int_{\mathbb{R}} \int_0^\infty \phi(x+t, e^x)\, dt\, dx \\ &= \int_{\mathbb{R}} \int_x^\infty \phi(t, e^x)\, dt\, dx \\ &= \int_0^\infty \int_{\ln y}^\infty \frac{1}{y} \cdot \phi(x,y)\, dx\, dy, \end{aligned}$$

which fails to converge if ϕ *does not identically vanish on the* x*-axis.*

In view of composing a distribution with smooth functions, if we composed w with a translation, e.g., $\Phi^\star w = w(x - x_0)$, it is easy to verify that

$$\Phi^\star(v \star w) = \Phi^\star v \star w = v \star \Phi^\star w.$$

2.2.5 The Fourier transform

The Fourier transform of a compactly-supported distribution u is defined as

$$\hat{u}(\vec{\xi}) = u_x\left(e^{-ix\cdot\vec{\xi}}\right). \tag{2.2.7}$$

The right-hand side in fact is defined as an entire function of $\vec{\xi} \in \mathbb{C}^n$, and is known as the Fourier-Laplace transform of u (Hörmander Theorem 7.1.14). If u is compactly-supported with order k, then

$$\begin{aligned} \left|\hat{u}\left(\vec{\xi}\right)\right| &\leq C_1 \left\|e^{ix\cdot\vec{\xi}}\right\|_{\mathcal{W}^{k,\infty}(\mathbb{R}^n)} \\ &\leq C_2\left(1 + \|\vec{\xi}\|\right)^k, \end{aligned} \tag{2.2.8}$$

for some constants C_1 and C_2. This is quite a weakening of the Riemann-Lebesgue lemma regarding the decay of Fourier transforms of $\mathcal{L}^1$ functions at infinity. As can be expected, the Fourier transform of compactly-supported distributions satisfies many of the properties of the Fourier transform operator on $\mathcal{L}^2(\mathbb{R}^n)$, including the convolution theorem, as seen by

$$\begin{aligned}
\widehat{u \star w}\left(\vec{\xi}\right) &= (u \star w)_x \left(e^{-ix\cdot\vec{\xi}}\right) \\
&= u_x \left(w_y \left(e^{-i(x+y)\cdot\vec{\xi}}\right)\right) \\
&= u_x \left(e^{-ix\cdot\vec{\xi}} w_y \left(e^{-iy\cdot\vec{\xi}}\right)\right) \\
&= u_x \left(e^{-ix\cdot\vec{\xi}}\right) w_y \left(e^{-iy\cdot\vec{\xi}}\right) \\
&= \hat{u}\left(\vec{\xi}\right) \hat{w}\left(\vec{\xi}\right),
\end{aligned}$$

for $u, w \in \mathcal{E}'(\mathbb{R}^n)$.

Inversion must be done in a distributional sense, however. Since the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to itself, we can define the Fourier transform on tempered distributions in a distributional sense by

$$\hat{u}(\phi) = u(\hat{\phi}), \quad u \in \mathcal{S}'(\mathbb{R}^n), \phi \in \mathcal{S}(\mathbb{R}^n),$$

which is compatible with (2.2.7) when $u \in \mathcal{E}'(\mathbb{R}^n)$. The inverse Fourier transform is thusly defined in a similar manner, and can be applied to $\hat{u}$.

A feature of the Fourier transform which will prove useful later is the following symmetry result:

Theorem 2.2.5.1. *If $w \in \mathcal{E}'(\mathbb{R}^n)$ has odd symmetry across a hyperplane $\{x \cdot \vec{\mathbf{v}} = t_0\}$ for some fixed $\vec{\mathbf{v}} \in \mathcal{S}^{n-1}$ and $t_0 \in \mathbb{R}^n$, i.e.,*

$$w_x(\phi(\bar{x} + (2t_0 - t)\vec{\mathbf{v}})) = -w(\phi), \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n),$$

where we write $x = \bar{x} + t\vec{\mathbf{v}}$ for $\bar{x} \in \vec{\mathbf{v}}^\perp$ and $t \in \mathbb{R}$, then $\hat{w}$ vanishes on $\vec{\mathbf{v}}^\perp$.

We will omit the proof of this result.

2.3 Wavefront Sets

The idea of the wavefront set is to not only describe the locations of singularities of $u \in \mathcal{D}'(X)$, but also the directions of these singularities. If $x_0 \notin \operatorname{sing\,supp}(u)$, then one can choose an open neighborhood U of x_0 for which u is $\mathcal{C}^\infty$ on U, and then choose $\phi \in \mathcal{C}_0^\infty(U)$ with $\phi(x_0) \neq 0$. It would then follow that $\phi u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and as such, $\widehat{\phi u}$ rapidly decays on $\mathbb{R}^n$ – for each $N \geq 0$, there is a $C_N \geq 0$ such that:

$$\left|\widehat{\phi u}\left(\vec{\xi}\right)\right| \leq C_N \left(1 + \|\vec{\xi}\|^2\right)^{-N/2}, \quad \vec{\xi} \in \mathbb{R}^n. \tag{2.3.1}$$

If $x \in \operatorname{sing\,supp}(u)$, then the above estimate fails for all choices of $\phi \in \mathcal{C}_0^\infty(X)$ where $\phi(x_0) \neq 0$. The wavefront set describes on which conic subset the estimate fails for at least one value of N.

Definition 2.3.0.1 (Wavefront set)**.** *Let $u \in \mathcal{D}'(X)$. For $x_0 \in X$ and nonzero $\vec{\xi}_0 \in \mathbb{R}^n$, we say $\left(x_0, \vec{\xi}_0\right) \notin WF(u)$ if there exists a $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ and open conic (closed under positive scaling) neighborhood Γ of $\vec{\xi}_0$ such that the estimate* (2.3.1) *is valid for $\vec{\xi} \in \Gamma$, for all N.*

This defines $WF(u)$ as a subset of $X \times \mathbb{R}^n \backslash 0$. Alternately, $\left(x_0, \vec{\xi}_0\right) \in WF(u)$ if for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ and open conic neighborhood Γ of $\vec{\xi}_0$, the estimate (2.3.1) fails on Γ for some N.

An alternate characterization of the wavefront set is that $\left(x_0, \vec{\xi}_0\right) \in WF(u)$ if and only if for every open neighborhood U of x_0 and open conic neighborhood Γ of $\vec{\xi}_0$, there exists $\phi \in \mathcal{C}_0^\infty(U)$ such that (2.3.1) fails on Γ for some N.

Some initial results following from this definition are that

$$WF(\psi u) \subseteq \operatorname{supp}\psi \times \mathbb{R}^n \,\cap\, WF(u),$$

$$WF(u \pm w) \subseteq WF(u) \,\cup\, WF(w),$$

whenever $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $u, w \in \mathcal{D}'(\mathbb{R}^n)$. The estimate (2.2.8) is essential in proving the latter of these two results.

Since the definition of conic is closure with respect to positive scaling, it follows from this definition that wavefront set of a distribution is invariant under positive scaling of the second component. However, if a distribution is real (in the sense of mapping real-valued test functions into $\mathbb{R}$), then we have invariance with respect to negative scaling as well.

Of greater interest is how partial differential operators and their inverses (when they exist) alter wavefront sets. Indeed, when P is a linear partial differential operator with coefficients in $\mathcal{C}^{\infty}$, then $WF(Pu) \subseteq WF(u)$, Hörmander formula (8.1.11), a corollary from the following result:

Theorem 2.3.0.2. *For $u \in \mathcal{D}'(X)$, $WF(\mathcal{D}_{\vec{\mathbf{v}}}u) \subseteq WF(u)$.*

Proof. Suppose $\left(x_0, \vec{\xi}_0\right) \notin WF(u)$. Then there is an open neighborhood U of x_0 such that $\widehat{\psi u}$ is rapidly decaying in some open conic neighborhood of $\vec{\xi}_0$ for all $\psi \in \mathcal{C}_0^{\infty}(U)$, In particular, choose ψ to be equal to one on some smaller open neighborhood V of x_0. Now choose $\phi \in \mathcal{C}_0^{\infty}(V)$ with $\phi(x_0) \neq 0$. Then

$$\phi \mathcal{D}_{\vec{\mathbf{v}}}(\psi u) = \phi \psi \mathcal{D}_{\vec{\mathbf{v}}} u,$$

and so we then have

$$\mathcal{F}(\phi \psi \mathcal{D}_{\vec{\mathbf{v}}} u) = \hat{\phi} \star \left\{ i\vec{\mathbf{v}} \cdot \vec{\xi}\, \widehat{\psi u} \right\}.$$

Since $\widehat{\psi u}$ rapidly decays on open conic neighborhood of $\vec{\xi}_0$, so must $i\vec{\mathbf{v}} \cdot \vec{\xi}\, \widehat{\psi u}$, and likewise, so must the convolution $\hat{\phi} \star \left\{ i\vec{\mathbf{v}} \cdot \vec{\xi}\, \widehat{\psi u} \right\}$. ❏

While the above result indicates that linear partial differential operators do not add singularities, the following result describes their capacity to remove them:

Theorem 2.3.0.3 (Microlocal property[1])**.** *If P is a differential operator of order m with $\mathcal{C}^{\infty}$ coefficients on a manifold X, then*

$$WF(u) \subseteq \operatorname{Char} P \,\cup\, WF(Pu), \quad u \in \mathcal{D}'(X),$$

where the characteristic set $\operatorname{Char} P$ *is defined by*

$$\operatorname{Char} P = \left\{ \left(x, \vec{\xi}\right) \in T^{\star}(X) \,\middle|\, P_m\left(x, \vec{\xi}\right) = 0 \right\},$$

and P_m is the principal symbol of P.

If P is merely a directional derivative, i.e., $P = \mathcal{D}_{\vec{\mathbf{v}}}$, then $P_m\left(x, \vec{\xi}\right) = i\vec{\xi} \cdot \vec{\mathbf{v}}$, and so

$$\operatorname{Char} P = \mathbb{R}^n \times \vec{\mathbf{v}}^{\perp}.$$

Hence, $\mathcal{D}_{\vec{\mathbf{v}}}$ may remove elements from the wavefront set of a distribution u whose second components are orthogonal to the direction of differentiation.

As for antidifferentiation, we will want to make use of the following theorem:

Theorem 2.3.0.4 (Hörmander 8.1.5)**.** *Let V be a linear subspace of $\mathbb{R}^n$ and $u = u_0\, dS$, where $u_0 \in \mathcal{C}^{\infty}(V)$ and dS is Euclidean surface measure (on V). That is,*

$$u(\phi) = \int_V u_0(x)\, \phi(x)\, dS(x).$$

Then

$$WF(u) \subseteq \operatorname{supp}(u) \times \left(V^{\perp} \backslash 0\right).$$

Proof. If $\chi \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$, then

$$\begin{aligned}
\mathcal{F}_{\mathbb{R}^n}(\chi u)\left(\vec{\xi} + \vec{\eta}\right) &= \int_V \chi(x)\, u_0(x)\, e^{-ix\cdot\left(\vec{\xi}+\vec{\eta}\right)}\, dS(x) \\
&= \int_V \chi(x)\, u_0(x)\, e^{-ix\cdot\vec{\xi}}\, dS(x) \\
&= \mathcal{F}_V(\chi u_0)\left(\vec{\xi}\right),
\end{aligned}$$

for $\vec{\xi} \in V$, $\vec{\eta} \in V^{\perp}$. Since $\chi u_0 \in \mathcal{C}_0^{\infty}(V)$, $\mathcal{F}_V(\chi u_0)$ decays rapidly in $\vec{\xi}$. Thus, if Γ is an open cone in $\mathbb{R}^n$, $\chi|_V \neq 0$, and $\int_V \chi u_0\, dS \neq 0$, then $\mathcal{F}_{\mathbb{R}^n}(\chi u)$ will rapidly decay on Γ if and only if $\Gamma \cap V = \emptyset$. ❑

The consequence of this result is the following:

Theorem 2.3.0.5. *The wavefront set of $J_{\vec{\mathbf{v}}}$ is given by*

$$WF(J_{\vec{\mathbf{v}}}) = \{0\} \times (\mathbb{R}^n \backslash 0) \,\cup\, \mathbb{R}^+\vec{\mathbf{v}} \times \left(\vec{\mathbf{v}}^{\perp} \backslash 0\right). \tag{2.3.2}$$

Proof. We note that $J_{\vec{\mathbf{v}}}$ is equal to Euclidean line measure on $V = \mathbb{R}\vec{\mathbf{v}}$ on the half-space $H = \{x \,|\, x \cdot \vec{\mathbf{v}} > 0\}$. Hence,

$$WF(J_{\vec{\mathbf{v}}}) \cap H \times \mathbb{R}^n = \mathbb{R}^+\vec{\mathbf{v}} \times \left(\vec{\mathbf{v}}^{\perp} \backslash 0\right).$$

Meanwhile, since $\delta_0 = \mathcal{D}_{\vec{\mathbf{v}}} J_{\vec{\mathbf{v}}}$, $\{0\} \times (\mathbb{R}^n \backslash 0) \subseteq WF(J_{\vec{\mathbf{v}}})$, and since the only point of $\operatorname{supp}(J_{\vec{\mathbf{v}}})$ outside of H is at the origin, all of the elements of $WF(J_{\vec{\mathbf{v}}})$ thus accounted for, and so

$$WF(J_{\vec{\mathbf{v}}}) = \{0\} \times (\mathbb{R}^n \backslash 0) \cup \mathbb{R}^+ \vec{\mathbf{v}} \times \left(\vec{\mathbf{v}}^\perp \backslash 0\right). \qquad \square$$

Since $J_{\vec{\mathbf{v}}}$ is a fundamental solution to $\mathcal{D}_{\vec{\mathbf{v}}}$, our interest in $J_{\vec{\mathbf{v}}}$ is in use as a convolution kernel, and so formula (8.2.16) from Hörmander will be necessary:

$$WF(k \star u) \subseteq \left\{ \left(x+y, \vec{\xi}\right) \,\middle|\, \left(x, \vec{\xi}\right) \in WF(k), \left(y, \vec{\xi}\right) \in WF(u) \right\}. \tag{2.3.3}$$

Applying this result to $J_{\vec{\mathbf{v}}}$, we obtain the following:

Theorem 2.3.0.6. *For $u \in \mathcal{D}'(\mathbb{R}^n)$ for which* $\operatorname{supp}(u) \cap (K - \mathbb{R}_{\geq 0}\vec{\mathbf{v}})$ *is compact for every compact set K,*

$$WF(\mathcal{J}_{\vec{\mathbf{v}}} u) \subseteq WF(u) \cup \left\{ \left(x + t\vec{\mathbf{v}}, \vec{\xi}\right) \,\middle|\, \left(x, \vec{\xi}\right) \in WF(u), \vec{\xi} \in \vec{\mathbf{v}}^\perp, t > 0 \right\}.$$

We will later give a demonstration of this result independent from Hörmander (8.2.16) in Chapter **??**.

It will become necessary to perform analysis in alternate coordinates in some circumstances. Recall that we define composition $\Phi^\star u$ when $u \in \mathcal{D}'(Y)$, $\Phi \in \mathcal{C}^\infty(X; Y)$, and $\mathcal{D}\Phi(x)$ is surjective for all $x \in X$. Using wavefront sets, it is possible to extend composition further. Suppose $\Gamma \subseteq X \times \mathbb{R}^n \backslash 0$ is closed, and conic in the second component. We then define the space

$$\mathcal{D}'_\Gamma(X) = \left\{ u \in \mathcal{D}'(X) \,\middle|\, WF(u) \subseteq \Gamma \right\},$$

with which we then assign a topology through application of the following result:

Proposition 2.3.0.1 (Hörmander 8.2.1)**.** *A distribution $u \in \mathcal{D}'(X)$ is in $\mathcal{D}'_\Gamma(X)$ if and only if for every $\phi \in \mathcal{C}_0^\infty(X)$ and closed cone $V \subseteq \mathbb{R}^n$ with*

$$\Gamma \cap \operatorname{supp}(\phi) \times V = \emptyset \tag{2.3.4}$$

we have

$$\sup_{\vec{\xi} \in V} \left\|\vec{\xi}\right\|^N \left|\widehat{\phi u}\left(\vec{\xi}\right)\right| < \infty, \quad N = 1, 2, \ldots.$$

We then define a topology on $\mathcal{D}'_\Gamma(X)$ in which we say

Definition 2.3.0.2 (Hörmander 8.2.2)**.** *For a sequence $u_j \in \mathcal{D}'_\Gamma(X)$, we shall say that $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$ if and only if*

$$u_j \to u \text{ in } \mathcal{D}'(X) \text{ (weakly)}$$

$$\lim_{j\to\infty} \sup_{\vec{\xi}\in V} \left\|\vec{\xi}\right\|^N \left|\widehat{\phi u}\left(\vec{\xi}\right) - \widehat{\phi u_j}\left(\vec{\xi}\right)\right| = 0,$$

for $N = 1, 2, \ldots$ if $\phi \in \mathcal{C}_0^\infty(X)$ and V is a closed cone in $\mathbb{R}^n$ such that (2.3.4) *is valid.*

Hormander Theorem 8.2.3 then implies that $\mathcal{C}_0^\infty(X)$ is dense in $\mathcal{D}'_\Gamma(X)$ with respect to this topology. We may now define composition of a distribution with a smooth function in a more general setting.

Theorem 2.3.0.7 (Hörmander 8.2.4)**.** *Let X and Y be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map. Denote the set of normals of the map by*

$$N_f = \left\{ (f(x), \vec{\eta}) \in Y \times \mathbb{R}^n \,\middle|\, \mathcal{D}f(x)^T \vec{\eta} = \vec{\mathbf{0}} \right\}. \tag{2.3.5}$$

Then the pullback $f^\star u$ can be defined in one and only one way for all $u \in \mathcal{D}'(Y)$ with

$$N_f \cap WF(u) = \emptyset \tag{2.3.6}$$

so that $f^\star u = u \circ f$ when $u \in \mathcal{C}^\infty$ and for any closed conic subset Γ of $Y \times \mathbb{R}^n \backslash 0$ with $\Gamma \cap N_f = \emptyset$ we have a continuous map $f^\star : \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'_{f^\star\Gamma}(X)$,

$$f^\star\Gamma = \left\{ \left(x, \mathcal{D}f(x)^T \vec{\eta}\right) \,\middle|\, (f(x), \vec{\eta}) \in \Gamma \right\}. \tag{2.3.7}$$

In particular we have for every $u \in \mathcal{D}'(Y)$ satisfying (2.3.6)

$$WF(f^\star u) \subseteq f^\star WF(u).$$

Note that if f is a diffeomorphism, then the above wavefront set inclusion can be replaced with equality.

Bibliography

[1] L. Hörmander. *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis.* Springer Verlag, Berlin, second edition, 1990.

[2] L. Schwartz. *Théorie des distributions.* Hermann, Paris, 1951.